# ON STRESSES IN AN ELASTIC HALF-PLANE

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The formulation and the solution in closed form of the first fundamental problem of the theory of elasticity for a half-plane are considered for the case when stresses at infinity may attain any finite values, while the principal vector of external loading applied to the boundary of the medium, generally speaking, is not finite. The solution is constructed by the method of Muskhelishvili [1].

**1.** Let the region $S$ occupied by the elastic medium represent the lower half-plane $y < 0$ of the plane of the variable $z = x + iy$. The boundary of the region, the real $x$-axis, will be designated by $L$. The problem consists in determining the state of stress in the elastic half-plane $S$ for given boundary external loadings.

With regard to the components of stress, it will be assumed that they are bounded in the whole region $S$. More precisely, we will require that these components of stress approach definite finite limits which are in general different from zero as $z$ approaches infinity along an arbitrary path, remaining within $S$. The rotation at infinity is assumed to be equal to zero.

Let the boundary conditions be given in the form (see [1], p. 346)

$$N(t) = a + f_1(t), \qquad T(t) = b + f_2(t) \qquad (Y_y = N,\ X_y = T \text{ on } L) \tag{1}$$

Here $N$ and $T$ are the normal and shear components of the external loading acting on the boundary of the medium, $a$, $b$ are the given constants, and $f_1$, $f_2$ are continuous functions of the abscissa $t$ given on $L$, satisfying for large $|t|$ the conditions

$$f_1(t) = o(t^{-1}), \qquad f_2(t) = o(t^{-1}) \tag{2}$$

As it will be seen in the following, the specification of the boundary condition (1) only is not sufficient for a unique determination of the state of stress. More precisely, for the determinacy of the problem it is necessary, in addition to the quantities $a$ and $b$, which represent the values of the components of stress $Y_y$ and $X_y$ at infinity respectively, to prescribe also the value of the stress $X_x$ at infinity

$$\lim X_x = c \quad \text{for } |z| \to \infty \tag{3}$$

Let us assume that the quantity $c$ is given. Let

$$Y = \int_L f_1(t)\,dt, \qquad X = \int_L f_2(t)\,dt \tag{4}$$

We now require that the sought holomorphic functions $\Phi(z)$ and $\Psi(z)$ be represented for large $|z|$ in the form

$$\Phi(z) = \Gamma - \frac{X+iY}{2\pi z} + o(z^{-1}), \qquad \Psi(z) = \Gamma' + \frac{X-iY}{2\pi z} + o(z^{-1}) \tag{5}$$

$$\Phi'(z) = \frac{X+iY}{2\pi z^2} + o(z^{-2}), \qquad \Gamma = \frac{a+c}{4}, \qquad \Gamma' = \frac{1}{2}(a-c) + ib$$

We supplement these conditions by the following ones

$$\varphi(z) = \Gamma z - \frac{X+iY}{2\pi}\ln z + o(1) + \text{const}, \qquad \psi(z) = \Gamma' z + \frac{X-iY}{2\pi}\ln z + o(1) + \text{const} \tag{6}$$

It is not difficult to prove that under the conditions formulated above, the problem yields a unique solution.

*Remark.* In the well known book by Timoshenko, as an application of a general representation of Airy function in the form of infinite series of partial solutions of the bi-harmonic equation in a circular ring, the problem is considered regarding the stresses in the wedge, which is bounded by the lines $\theta = 0$ and $\theta = \beta$ ($\theta$ is the polar angle of the point $z = re^{i\theta}$, $0 < \beta < 2\pi$), with the boundary conditions ([2], Section 37)

$$N = -q, \quad T = 0 \quad \text{for } \vartheta = 0, \qquad N = T = 0 \quad \text{for } \vartheta = \beta \tag{7}$$

where $q$ is a given constant. No other limitations, besides (7) are imposed upon the sought state of stress. The solution of the problem is sought in the form

$$\Phi(z) = iC\ln z + B, \qquad \Psi(z) = B' + iC' \tag{8}$$

where $B$, $C$, $B'$, $C'$ are real constants to be determined.

The substitution of (8) into the boundary conditions (7) yields a system of linear equations with respect to the sought constants, from which they may be uniquely determined, except in the cases $\beta = \pi$ (the case of the half-plane) and $\beta - \operatorname{tg}\beta = 0$.

In the two indicated cases a verification shows, the determinant of the system vanishes.

In the case $\beta = \pi$ together with the solution (([2], p. 139. formula [d]),

$$\Phi(z) = \frac{q}{2\pi i}\ln z - \frac{q}{2}, \qquad \Psi(z) = -\frac{q}{2\pi i}$$

the problem admits also multiple solutions given by the formulas

$$\Phi(z) = \frac{q}{2\pi i}\ln z - \frac{q}{2} + A, \qquad \Psi(z) = -\frac{q}{2\pi i} - 2A \tag{9}$$

where $A$ is an arbitrary real constant. It is easily seen that the quantity $A$ influences only the component of stress $X_x$, which therefore must be subjected to some additional condition in order to insure uniqueness.

As regards the case $\beta - \operatorname{tg}\beta = 0$, here the solutions of type (8) do not exist at all. More precisely, the problem can be solved in terms of functions (8) only if $q = 0$, and the trivial solution has the form:

$$\Phi(z) = A\left(i\ln z + \tfrac{1}{2}\operatorname{tg}\beta\right), \qquad \Psi(z) = -A(i + \operatorname{tg}\beta)$$

where $A$ is an arbitrary real constant.

2. In solving the formulated problem we shall follow Muskhelishvili, who indicated a simple solution in the case when $a = b = c = 0$ ([1], Section 93).

On the boundary of the half-plane $S$, as is known, we shall have

$$\Phi(t) + \overline{\Phi(t)} + t\overline{\Phi'(t)} + \overline{\Psi(t)} = N - iT \tag{10}$$

Applying to both sides of the preceding equation the operation

$$\frac{1}{2\pi i}\int_L \frac{dt}{t - z}$$

where $z$ is an arbitrary point on the region $S$, taking into account conditions (1), (5), and using the known properties of the Cauchy-type integral along an infinite line ([1], Section 72), we obtain

$$\Phi(z) = \Gamma - \frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t - z}, \qquad f(t) = f_1(t) - if_2(t) \tag{11}$$

Having determined $\Phi(z)$, the function $\Psi(z)$ may be found in an analogous manner by means of a passage to conjugate expressions, starting from the boundary conditions obtainable from (10). Taking (11) into account, we obtain

$$\Psi(z) = \Gamma + \Gamma' - \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t - z} - \Phi(z) - z\Phi'(z) = \Gamma' - \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t - z} + \frac{1}{2\pi i}\int_L \frac{f(t)\,t\,dt}{(t - z)^2} \tag{12}$$

On the basis of known properties of integrals of the type (11) ([1], Section 71) we conclude immediately that if the function $f'(t)$ satisfies Hoelder's condition on an arbitrary finite section of the straight line $L$, and the derivatives $tf(t)$, $t^2 f'(t)$ also satisfy Hoelder's condition in the neighborhood of the infinite point, then the functions $\Phi(z)$, $\Psi(z)$ found above will satisfy all conditions of our problem. In particular, the functions $\Phi$, $\Phi'$, $\Psi$ for large values of $|z|$ will be of the form to be found from (5). The problem is solved.

## BIBLIOGRAPHY

1. Muskhelishvili, N.I., *Nekotorye osnovnye zadachi matematicheskoi teorii uprugosti* (*Some Basic Problems of the Mathematical Theory of Elasticity*). Izdatel'stvo Akademii Nauk SSSR, Moscow, 1949.

2. Timoshenko, S.P., *Teoriia uprugosti* (per. s angliiskogo). (*Theory of Elasticity*). Gostekhizdat, 1934.

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